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Ricci and Maxwell collineations in a null electromagnetic field

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Abstract. It has been shown that Synge's metric representing null electromagnetic field admits proper Ricci collineation. The space–time admits curvature collineation in special cases and the CC vectors are obtainable from the RC vector by an appropriate choice of the arbitrary quantities involved. When the RC vector is constrained to give motion the space–time permits 5-, 6- and 7-parameter groups of motions in the different cases considered. It is found that the space–time admits Maxwell and Ricci collineations along common directions in special cases for which both the conservation laws are satisfied.

1. The metric and electromagnetic field

Synge's (1960) metric plays an important part in demonstrating plane gravitational waves in the case of empty space. In this paper we show that the metric represents null electromagnetic field under certain restrictions when $R_{ij} \neq 0$. Electromagnetic field equations depend upon the Ricci and electromagnetic field tensors. Hence it is interesting to consider in this case whether the space–time admits Ricci and Maxwell collineations and also to see whether vector fields exist along which both the collineations are permissible together. It is found that the space–time admits both Ricci and Maxwell collineations and the collineation vectors have been determined separately. It is shown that in special cases the two exist along the same vector field and the conservation laws couched in terms of collineation vectors are satisfied.

The metric under consideration is (Synge 1960)

$$ds^2 = e^{2P}(dx^1)^2 + e^{2Q}(dx^2)^2 + 2\,dx^3\,dx^4. \tag{1.1}$$

It is easy to see that both $[P_{,44} + (P_{,4})^2] \equiv p$ and $[Q_{,44} + (Q_{,4})^2] \equiv q$ cannot be zero or the space–time degenerates into flat space. (In this paper a comma and a semicolon followed by any suffix or suffixes denote partial and covariant differentiation with respect to the corresponding coordinates.)

For equation (1.1), $R_{ij} \neq 0$, $R = 0$ are satisfied and the Ricci tensor can be written as

$$R_{ij} = \sigma l_i l_j \tag{1.2}$$

where $\sigma = [p+q]$ and $l_i = (0, 0, 0, 1)$ is the null vector.

For a gravitational radiation field the Riemann tensor bears a special relationship to the null cone. One such relationship is

$$(R_{hijk} + \mathrm{i}\,{}^*R_{hijk})l^k = 0 \tag{1.3}$$

where l^k is a null vector and ${}^*R_{hijk}$ is the dual tensor of R_{hijk}. Equation (1.3) imposes a very severe condition on the Riemann tensor which is satisfied only asymptotically in the wave zone of a radiating system and exactly in a space–time representing plane gravitational waves and a few other special cases. It can be seen that equation (1.3) is satisfied for equation (1.1). Hence the space–time under consideration represents a gravitational radiation field possibly associated with an electromagnetic field.

To study the electromagnetic field represented by the space–time (1.1), we consider source-free Einstein–Maxwell equations, namely

$$R_{ij}-\tfrac{1}{2}Rg_{ij} = -kE_{ij} \tag{1.4}$$

where $k = 8\pi$, $4\pi E_{ij} = -F_{ik}F_{j.}^{k}+\frac{1}{4}g_{ij}F_{ab}F^{ab}$ where F_{ij} is the electromagnetic field tensor. Obviously the field equation (1.4) reduces to

$$R_{ij} = 2F_{ik}F_{j.}^{k} \tag{1.5}$$

as $F_{ij}F^{ij} = 0$ in view of equation (1.2).

Using equation (1.5) and the components of the Ricci tensor R_{ij} for (1.1) we get the following:

$$F_{12} = F_{23} = F_{34} = F_{31} = 0 \tag{1.6}$$

$$[g^{11}(F_{14})^2+g^{22}(F_{24})^2] = \tfrac{1}{2}(p+q). \tag{1.7}$$

From equations (1.6) and (1.7) we find that the Maxwell equations

$$F^{ij}{}_{;j} = 0 \qquad F_{ij;k}+F_{jk;i}+F_{ki;j} = 0 \tag{1.8}$$

are satisfied.

Hence from the Einstein–Maxwell equation (1.5) we conclude that in the space–time given by equation (1.1), the only non-vanishing components of the electromagnetic field tensor are in general F_{14} and F_{24} satisfying the Maxwell equations and also

$$F_{ij}F^{ij} = 0 \qquad F_{ij}{}^{*}F^{ij} = 0$$

showing that the space–time represents a null electromagnetic radiation field.

2. Ricci collineation and curvature collineation

A space–time is said to permit Ricci collineation (Katzin *et al* 1969) along a field vector ξ^i if

$$\mathscr{L}_{\xi}R_{ij} = 0 \tag{2.1}$$

where $\mathscr{L}_{\xi}$ stands for the Lie derivative along the field vector ξ^i.

Here we examine whether the space–time under consideration permits Ricci collineation (RC). Using the components of R_{ij} for equation (1.1) in equation (2.1) we get the following independent equations:

$$\xi^4_{,1} = 0 = \xi^4_{,2} = \xi^4_{,3}$$

and

$$R_{44,4}\xi^4+2R_{44}\xi^4_{,4} = 0. \tag{2.2}$$

The integration of equation (2.2) gives the components of the Ricci collineation vector as

$$\xi^1 = \xi^1(x^1, x^2, x^3, x^4)$$
$$\xi^2 = \xi^2(x^1, x^2, x^3, x^4)$$
$$\xi^3 = \xi^3(x^1, x^2, x^3, x^4)$$
$$\xi^4 = K/(p+q)^{1/2} \tag{2.3}$$

where K is an arbitrary constant of integration.

To determine whether the RC is proper we proceed to examine curvature collineation (CC) and other higher symmetries for the space–time. CC always implies RC for a space–time and for the RC to be proper the space–time does not in general permit CC.

A space–time is said to admit curvature collineation (Katzin *et al* 1969) along a field vector ξ^i if

$$\mathscr{L}_\xi R^{h}_{\cdot ijk} = 0. \tag{2.4}$$

Clearly this gives 96 equations.

From equation (2.4) it can be seen that three possible cases arise when $p \neq 0$, $q \neq 0$ which are listed below:

Case (*a*) $\quad p = q$ and $\xi^1_{,2}\, e^{2P} = -\xi^2_{,1}\, e^{2Q}$ (2.5)

Case (*b*) $\quad p = q$ and $\xi^1_{,2} = 0 = \xi^2_{,1}$ (2.6)

Case (*c*) $\quad p \neq q$ and $\xi^1_{,2} = 0 = \xi^2_{,1}$. (2.7)

In order to determine the components of the CC vector in case (*a*) we find on solving equation (2.4) for the space–time that

$$\xi^1 = \psi x^1 + a_1\, e^{-(P-Q)}x^2 + \psi_1 \tag{2.8}$$

$$\xi^2 = \left[\psi + K_1\left(\frac{P_{,4} - Q_{,4}}{p^{1/2}}\right)\right]x^2 - a_1\, e^{(P-Q)}x^1 + \psi_2 \tag{2.9}$$

$$\xi^3 = -e^{2P}\left[\psi_{,4}\frac{(x^1)^2}{2} + \psi_{1,4}x^1\right] - e^{2Q}\left\{\left[\psi + K_1\left(\frac{P_{,4} - Q_{,4}}{p^{1/2}}\right)\right]_{,4}\frac{(x^2)^2}{2} + \psi_{2,4}x^2\right\}$$
$$+ a_1(P_{,4} - Q_{,4})\, e^{(P+Q)}x^1x^2 + \left[2\psi + \frac{K_1}{2}\left(\frac{4P_{,4}p + p_{,4}}{p^{3/2}}\right)\right]x^3 + \psi_3 \tag{2.10}$$

and

$$\xi^4 = \frac{K_1}{p^{1/2}} = \frac{K_2}{q^{1/2}} \tag{2.11}$$

where $K_1^2 = K_2^2$ and K_1, K_2, a_1 are arbitrary constants and $\psi, \psi_1, \psi_2, \psi_3$ are arbitrary functions of x^4 alone. Hence the CC vector in this case is given by the equations (2.8)–(2.11).

Considering the case (*b*) it is obvious that the components of the CC vector in this case are also given by equations (2.8)–(2.11) with the condition that $a_1 = 0$ and $K_1^2 = K_2^2$.

For the case (*c*) it may be noted that the components of the CC vector are once again given by the set of equations (2.8)–(2.11) when $a_1 = 0$ and $K_1^2 q = K_2^2 p$.

Similarly the other two cases when $p \neq 0, q = 0$ and $p = 0, q \neq 0$ can be worked out.

Hence we conclude that the space–time (1.1) admits curvature collineation in all possible cases as is expected and the CC vector in different cases can be deduced from the RC vector by choosing appropriate ξ^1, ξ^2, ξ^3 and K in equation (2.3). Similarly it can be seen that RC vector (2.3) does not give motion in general and other higher symmetries such as conformal motion, homothetic motion, conformal collineation, affine collineation, projective collineation etc because of the general character of its first three components. Hence the RC vector (2.3) is proper.

3. Ricci collineation and motion

The vector field ξ^i given by equation (2.3) defines Ricci collineation for the space–time (1.1) when $R_{ij} \neq 0$. A study of this vector when constrained to give motion is of interest.

A space–time V_4 is said to admit motion if there exists a vector ξ^i satisfying

$$\mathscr{L}_\xi g_{ij} = 0. \tag{3.1}$$

The vector ξ^i is known as the Killing vector if equation (3.1) is satisfied.

Solving the equations (3.1) we get on integration

$$\xi^1 = -K\left(\frac{P_{,4}}{\sigma^{1/2}}\right)x^1 + a_1\, e^{-(P-Q)}x^2 + a_2 \int e^{-2P}\, dx^4 + a_3 \tag{3.2}$$

$$\xi^2 = -K\left(\frac{Q_{,4}}{\sigma^{1/2}}\right)x^2 - a_1\, e^{(P-Q)}x^1 + a_4 \int e^{-2Q}\, dx^4 + a_5 \tag{3.3}$$

$$\xi^3 = \left[K\, e^{2P}\left(\frac{P_{,4}}{\sigma^{1/2}}\right)_{,4}\right]\frac{(x^1)^2}{2} + \left[K\, e^{2Q}\left(\frac{Q_{,4}}{\sigma^{1/2}}\right)_{,4}\right]\frac{(x^2)^2}{2} + [a_1(P_{,4} - Q_{,4})\, e^{(P+Q)}]x^1x^2 - a_2x^1$$
$$- a_4x^2 + \frac{K}{2}\left(\frac{\sigma_{,4}}{\sigma^{3/2}}\right)x^3 + a_6 \tag{3.4}$$

with the conditions

$$K\left[e^{2P}\left(\frac{P_{,4}}{\sigma^{1/2}}\right)_{,4}\right]_{,4} = 0$$

$$K\left[e^{2Q}\left(\frac{Q_{,4}}{\sigma^{1/2}}\right)_{,4}\right]_{,4} = 0$$

$$a_1[(P_{,4} - Q_{,4})\, e^{(P+Q)}]_{,4} = 0$$

and

$$K\left[\frac{\sigma_{,4}}{\sigma^{3/2}}\right]_{,4} = 0. \tag{3.5}$$

Also equation (2.3) gives

$$\xi^4 = K/\sigma^{1/2}. \tag{3.6}$$

The space–time V_4 therefore admits motion if the conditions (3.5) are satisfied in which case the RC vector reduces to the Killing vector whose components are given by equations (3.2)–(3.4) and (3.6). Considering different cases we find:

Case (i). P and Q arbitrary and hence $K = 0$, $a_1 = 0$. In this case the space–time admits a 5-parameter group of motions.

Case (ii). $K = 0$, $a_1 \neq 0$ and hence $p = q$. The space–time admits a 6-parameter group of motions.

Case (iii). $K \neq 0$, $a_1 \neq 0$ and hence $p = q = \frac{1}{2}(ax^4+b)^{-2}$ where a, b are arbitrary constants. Here the space–time admits a 7-parameter group of motions.

Singh and Sharma (1975) have already shown that in the empty case 5- and 6-parameter groups of motions are admissible.

4. Maxwell collineation

Electromagnetic fields are said to admit Maxwell collineation (Collinson 1970) along the vector field ξ^i if

$$\mathscr{L}_\xi F^i_{\cdot j} = 0 \tag{4.1}$$

in which case the vector field ξ^i is called the MC vector.

We have seen in § 1 that the non-empty space–time (1.1) admits null electromagnetic field satisfying the Maxwell equations, and also RC along the vector field ξ^i given by equation (2.3). It is of interest to examine whether the space–time admits MC. In § 1 we found that the only surviving components of the electromagnetic field tensor can be F_{14} and F_{24}. Also from equation (1.7) it is clear that F_{14} and F_{24} cannot be determined uniquely. Considering different possibilities three cases arise:

Case (i). $F_{14} = 0, F_{24} \neq 0$.

Case (ii). $F_{24} = 0, F_{14} \neq 0$.

Case (iii). $F_{14} \neq 0, F_{24} \neq 0$.

For case (i) the set of equations (4.1) give on integration

$$\xi^1 = G_1(x^1, x^4) \tag{4.2}$$

$$\xi^2 = Gx^2 + G_2 \tag{4.3}$$

$$\xi^3 = G_3(x^1, x^4) - e^{2Q}[\tfrac{1}{2}G_{,4}(x^2)^2 + G_{2,4}x^2] + \left[G + \frac{(e^Q\sigma^{1/2})_{,4}}{e^Q\sigma^{1/2}}G_4\right]x^3 \tag{4.4}$$

$$\xi^4 = G_4(x^4) \tag{4.5}$$

where G_1, G_3 are functions of x^1, x^4 and G_2, G_4 are functions of x^4 alone and

$$G = \frac{(G_4 e^Q\sigma^{1/2})_{,4}}{e^Q\sigma^{1/2}} - 2Q_{,4}G_4.$$

Hence the components of the MC vector are given by equations (4.2)–(4.5) and the RC vector (2.3) in this case also reduces to the form given by these equations and

$$\xi^4 = K/\sigma^{1/2}. \tag{4.6}$$

Thus the vector field whose components are given by equations (4.2)–(4.4) and (4.6) defines RC as well as MC.

For the case (ii), considering (4.1) we determine

$$\xi^1 = Ux^1 + U_1 \tag{4.7}$$

$$\xi^2 = U_2(x^2, x^4) \tag{4.8}$$

$$\xi^3 = U_3(x^2, x^4) - e^{2P}[\tfrac{1}{2}U_{,4}(x^1)^2 + U_{1,4}x^1] + \left[U + \frac{(e^P\sigma^{1/2})_{,4}}{e^P\sigma^{1/2}}U_4\right]x^3 \tag{4.9}$$

$$\xi^4 = U_4 \tag{4.10}$$

where U_2, U_3 are functions of x^2, x^4 and U_1, U_4 are functions of x^4 alone and

$$U = \frac{(U_4 e^P\sigma^{1/2})_{,4}}{e^P\sigma^{1/2}} - 2P_{,4}U_4.$$

Hence the components of the vector field ξ^i given by equations (4.7)–(4.10) define Maxwell collineation for the space–time and it is easy to see that the RC vector (2.3) in this case reduces to the form given by equations (4.7)–(4.9) and (4.6). Thus the vector field whose components are given by these equations defines RC as well as MC for the space–time.

For case (iii) we assume $F_{14} = \alpha$, $F_{24} = \beta$ where α, β are non-zero arbitrary functions of x^4 satisfying equation (1.7). Using them in the set of equations (4.1) we get on simplification

$$\xi^1 = Ax^1 + \frac{g^{11}}{g^{22}\beta}[(\alpha v_4)_{,4} - \alpha(A + 2P_{,4}v_4)]x^2 + A_1 \tag{4.11}$$

$$\xi^2 = \frac{g^{22}}{g^{11}\alpha}[(\beta v_4)_{,4} - \beta(v + 2Q_{,4}v_4)]x^1 + vx^2 + A_2 \tag{4.12}$$

$$\xi^3 = \left[v_1x^1 - \frac{1}{2}\frac{a_1}{g^{11}\alpha}(x^1)^2\right] + \left[v_2x^2 - \frac{1}{2}\frac{a_2}{g^{22}\beta}(x^2)^2\right] + \frac{1}{\beta}\left\{\beta_{,4}v_4 + \beta v + \frac{g^{11}\alpha}{g^{22}\beta}\right.$$
$$\left.\times[(\alpha v_4)_{,4} - \alpha(A + 2P_{,4}v_4)]\right\}x^3 + v_3 \tag{4.13}$$

$$\xi^4 = v_4(x^4) \tag{4.14}$$

where A, A_1, A_2, v_3, v_4 are arbitrary functions of x^4 only and

$$v = \frac{g^{11}\alpha[(\alpha v_4)_{,4} + v_4(\alpha_{,4} - 2\alpha P_{,4})] + g^{22}\beta[(\beta v_4)_{,4} + v_4(\beta_{,4} - 2\beta Q_{,4})]}{(g^{11}\alpha^2 + g^{22}\beta^2)}$$
$$- \{g^{22}\beta\beta_{,4}v_4 + \alpha g^{11}[(\alpha v_4)_{,4} - \alpha(A + 2P_{,4}v_4)]\}/g^{22}\beta^2$$

$$a_1 = \alpha A_{,4} + \beta\left\{\frac{g^{22}}{g^{11}\alpha}[(\beta v_4)_{,4} - \beta(v + 2Q_{,4}v_4)]\right\}_{,4}$$

$$a_2 = \beta v_{,4} + \alpha\left\{\frac{g^{11}}{g^{22}\beta}[(\alpha v_4)_{,4} - \alpha(A + 2P_{,4}v_4)]\right\}_{,4}$$

$$v_1 = \frac{a - (\alpha A_{1,4} + \beta A_{2,4})}{g^{11}\alpha} \qquad v_2 = -\frac{a}{g^{22}\beta}.$$

Here a is an arbitrary function of x^4 alone.

Hence the vector field whose components are given by equations (4.11)–(4.14) defines MC and by choosing ξ^4 as in equation (4.6), which is a particular case of equation (4.14), this vector field gives a family of vectors for which the space–time admits RC as well.

5. Conclusion

Katzin *et al* (1969) derived the conservation law for the case $R = 0$, $R_{ij} \neq 0$ in terms of a vector field ξ^i given by $\mathscr{L}_\xi R_{ij} = 0$. Collinson (1970) generalized the result to the case when $R \neq 0$. He also derived the conservation law in terms of a vector field giving Maxwell collineation in a null electromagnetic field. Ordinarily the two conservation laws are satisfied along two different vector fields. In the present case we find that there is a common vector field giving Ricci as well as Maxwell collineation along which both the conservation laws are satisfied.

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